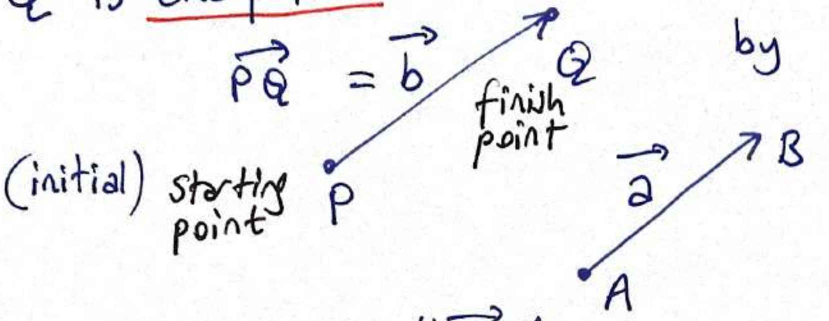
Calculus III

Ch. 9 9.1. Vectors in the plane and in space

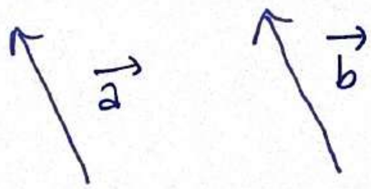
A Vector is a quantity (such as velocity or force) that has both → magnitude. → direction.

We will represent a vector as $\overrightarrow{PQ}$. P is starting point and Q is end point. Sometimes it is given by $\vec{a}, \vec{b}, \vec{c}$, etc.



The Vector $\overrightarrow{PQ}$ has magnitude $\|\overrightarrow{PQ}\|$.

Two vectors are equal if they have the same magnitude and direction.



$\vec{a} = \vec{b}$

A vector has magnitude 0 is called a zero (null) vector denoted by $\vec{0}$. It has no any direction. → $\|\vec{0}\| = 0$.

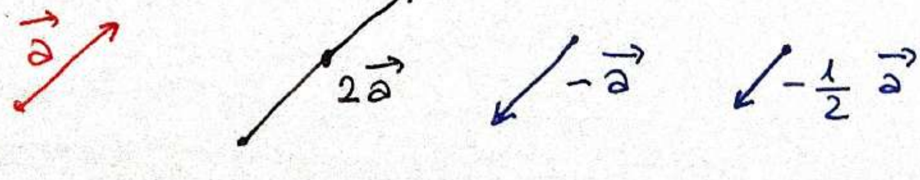
Vector Operations:

If $\vec{a} = m \cdot \vec{b}$ then $\vec{a}$ parallel to $\vec{b}$. $\boxed{\vec{a} \upuparrows \vec{b}}$

$m \in \mathbb{R}$ (scalar multiple)

- if $m > 0$, then $\vec{a} \upuparrows \vec{b}$, (parallel)
- if $m < 0$, then $\vec{a} \uparrow\downarrow \vec{b}$ (antiparallel) = (opposite direction)

" − " means



$\vec{u} = m\vec{v}$.

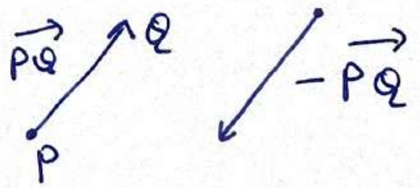
Ex. $\vec{u} = \vec{v}$ (are they parallel or opposite direction?)

Here, $m=1$ and $m>0$. Then $\vec{u} \uparrow\uparrow \vec{v}$.

Ex. $\vec{u} = -\frac{1}{2}\vec{v}$?

$m = -\frac{1}{2} < 0$ then $\vec{u} \uparrow\downarrow \vec{v}$.

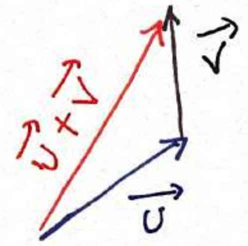
$\overrightarrow{PQ} = -\overrightarrow{QP}$ for any distinct points P and Q.



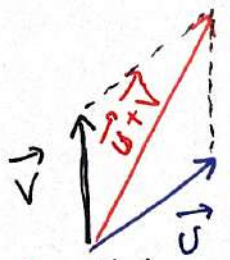
$\vec{0} = m.\vec{0}$ //

Force and velocity vectors can be added according to triangular rule (vector addition)

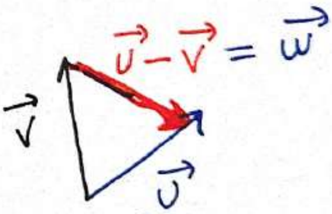
3 rules :



Triangular rule



Parallelogram rule



Difference rule

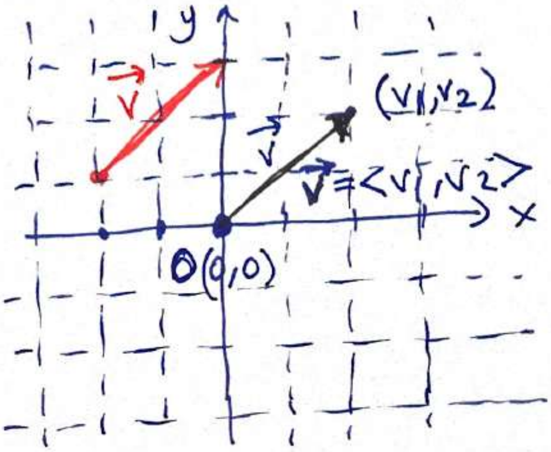
$\vec{w} + \vec{v} = \vec{u}$,

also $v + w = \vec{u}$.

$\left(\vec{v}+\vec{w} = \vec{w}+\vec{v} \text{ commutative} \right)$

component form of a vecter is $\vec{v} = \langle v_1, v_2 \rangle$.

angle brackets used for a vector in $\mathbb{R}^2$



v_1 : first component
v_2 : second component
} of a vector.

these are scalars.

$\langle \vec{v_1}, \vec{v_2} \rangle$:	a vector in $\mathbb{R}^2$
(v_1, v_2) :	a point in $\mathbb{R}^2$

If $A(a,b)$ and $B(c,d) \in \mathbb{R}^2$ then vector $\overrightarrow{AB}$ has the standart form :

points of $\mathbb{R}^2$

$$\boxed{\overrightarrow{AB} = \langle c-a, d-b \rangle} .$$

$\overrightarrow{AB} = B - A$

Vector operations for component form :

$\langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle \iff a_1 = b_1$ and $a_2 = b_2$.

iff
(if and only if)

$k \langle a_1, a_2 \rangle = \langle k \cdot a_1, k a_2 \rangle$

$k \in \mathbb{R}$
(constant)

$\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$

$\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$

Ex. For $\vec{u} = \langle 1, -3 \rangle$ and $\vec{v} = \langle 2, 4 \rangle$

find the following :

- $\vec{u} + \vec{v} = \langle 1, -3 \rangle + \langle 2, 4 \rangle = \langle 1+2, -3+4 \rangle = \langle 3, 1 \rangle$
- $2\vec{u} - 3\vec{v} = \langle 2, -6 \rangle - \langle 6, 12 \rangle = \langle 2-6, -6-12 \rangle$
  $= \langle -4, -18 \rangle$
- $\frac{5}{2}\vec{v} = \langle \frac{5}{2} \cdot 2, \frac{5}{2} \cdot 4 \rangle = \langle 5, 10 \rangle$
- $-\frac{2}{3}\vec{u} = \langle -\frac{2}{3} 1, -\frac{2}{3}(-3) \rangle = \langle -\frac{2}{3}, 2 \rangle$

Linear combination : $\underline{a\vec{u} + b\vec{v}}$, $a, b \in \mathbb{R}$

$$a\vec{u} + b\vec{v} = a\langle u_1, u_2 \rangle + b\langle v_1, v_2 \rangle$$
$$= \langle au_1, au_2 \rangle + \langle bv_1, bv_2 \rangle$$
$$= \langle au_1 + bv_1, au_2 + bv_2 \rangle .$$

$\forall \alpha, \beta \in \mathbb{R}$ and for $\forall$ vectors $\vec{u}, \vec{v}, \vec{w}$ in a plane :

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity)] for vector addition
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity)] for vector addition
- $(\alpha \cdot \beta)\vec{u} = \alpha \cdot (\beta \cdot \vec{u})$ (associativity of scalar multiplication)
- $\vec{u} + \vec{0} = \vec{u}$ (identity for addition)
- $\vec{u} + (-\vec{u}) = \vec{0}$ (Inverse property)
- $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$ (vector distributivity)
- $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ (scalar distributivity)

In Mathematics, some notations :

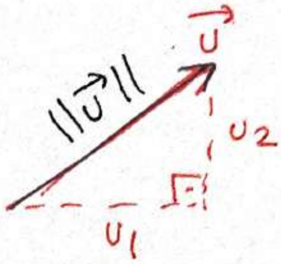
$\forall$: for all

$\exists$: there exists

v is shown in components form

When $v = \langle v_1, v_2 \rangle$ then its length is determined by
component form

$$\|v\| = \sqrt{v_1^2 + v_2^2}$$



Using Pythagorean theorem.

- If a vector has length 1, it is called Unit vector.

and a direction vector for a given non-zero vector $\vec{v}$ is a unit vector $\vec{u}$ that points in the same direction as $\vec{v}$.

Such a vector can be determined by simplify "dividing" $\vec{v}$ by its length $\|\vec{v}\|$ (or multiplying by the scalar $\frac{1}{\|\vec{v}\|}$); that is,

$$\boxed{\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}} \cdot \quad \|\vec{u}\| = 1.$$

Ex : Find the "direction vector" for the vector

$\vec{v} = \langle 3, -4 \rangle \longrightarrow$ angle bracket

$\|\vec{v}\| = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = 5$. Then, the direction
Magnitude

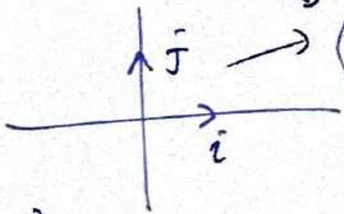
vector : $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 3, -4 \rangle}{5} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$. $\longrightarrow$ direction vector.

Check it!

$\Rightarrow \|\vec{u}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$ is a unit vector.

$\mathbb{R}^2$

$\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$ → standart base vectors

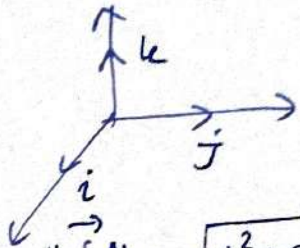


$\|\vec{i}\| = \sqrt{1^2+0^2} = 1$

$\|\vec{j}\| = \sqrt{0^2+1^2} = 1$

$\mathbb{R}^3$

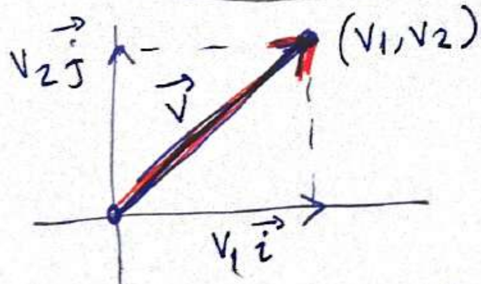
$$\left.\begin{aligned} \vec{i} &= \langle 1,0,0 \rangle \\ \vec{j} &= \langle 0,1,0 \rangle \\ \vec{k} &= \langle 0,0,1 \rangle \end{aligned}\right\} \to \text{standart base vectors}$$



$\|\vec{i}\| = \sqrt{1^2+0^2+0^2} = 1$

$\|\vec{j}\| = 1$, $\|\vec{k}\| = 1$.

$$\begin{aligned} \vec{v} &= \langle v_1, v_2 \rangle \\ &= v_1 \langle 1,0 \rangle + v_2 \langle 0,1 \rangle \\ &= v_1 \cdot \vec{i} + v_2 \cdot \vec{j} \end{aligned}$$



$\vec{v} = v_1 \vec{i} + v_2 \vec{j}$

standart representation of the vector $\vec{v}$

Ex: Find the standart representation of the vector $\overrightarrow{PQ}$, for the points $P(1,3)$, $Q(-2,4)$.

$\overrightarrow{PQ} = \langle -2-1, 4-3 \rangle = \langle -3, 1 \rangle$.

So, the standart representation of the vector $\overrightarrow{PQ}$ is determined by

$$\overrightarrow{PQ} = -3\vec{i} + 1\vec{j}.$$

Ex: If $\vec{u} = 2\vec{i} + 3\vec{j}$,
$\vec{v} = -3\vec{i} - 5\vec{j}$ and $\vec{w} = \vec{i} + 5\vec{j}$,
what is the standard representation of the vector: $-3\vec{u} + 2\vec{v} - 4\vec{w}$?

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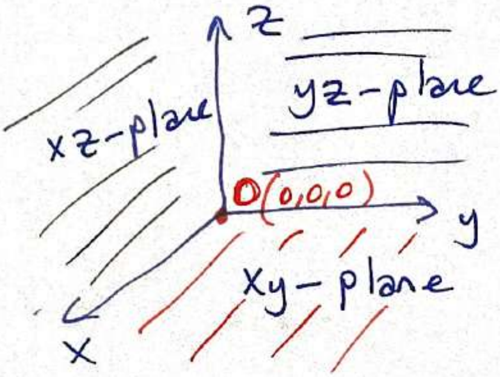
9.2. Coordinates and vectors in $\mathbb{R}^3$.

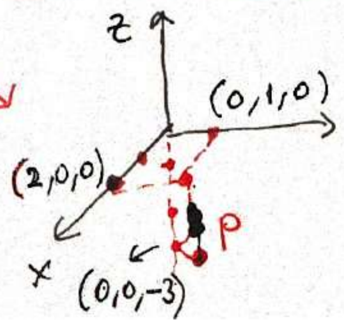


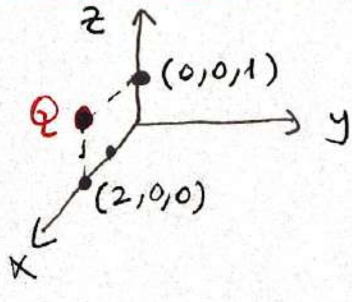
→ coordinate planes.

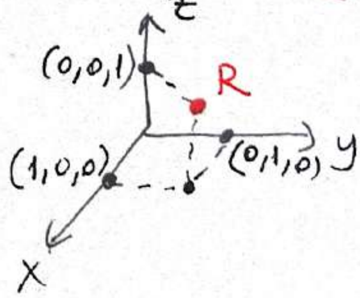
Ex: Graph the following (without technology)
→ P(2, 1, −3), Q(2, 0, 1), R(1, 1, 1).

The distance, in $\mathbb{R}^2$, from the origin $(0,0)$ to point (α, β) is
$$\boxed{d = \sqrt{\alpha^2 + \beta^2}},$$
and in $\mathbb{R}^3$, distance from $(0,0,0)$ to (α, β, γ) is $d = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$

For $P_1(\alpha_1, \beta_1, \gamma_1)$ and $P_2(\alpha_2, \beta_2, \gamma_2)$, the distance of two: (from P_1 to P_2) is
$$|P_1P_2| = \sqrt{(\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\gamma_2 - \gamma_1)^2}.$$







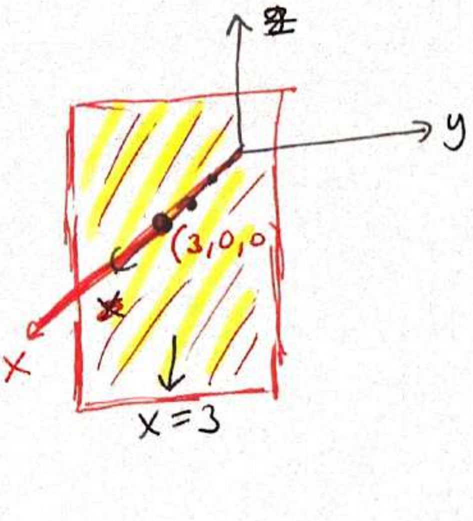
Ex: Find the distance between $(3,5,2)$ and $(-1,0,3)$.

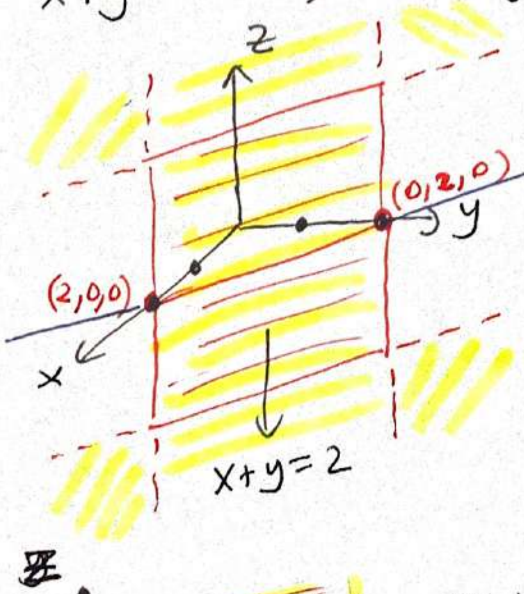
$$d=\sqrt{(-1-3)^2+(0-5)^2+(3-2)^2}$$
$$=\sqrt{(-4)^2+(-5)^2+1^2}$$
$$=\sqrt{16+25+1}$$
$$=\sqrt{42}.$$

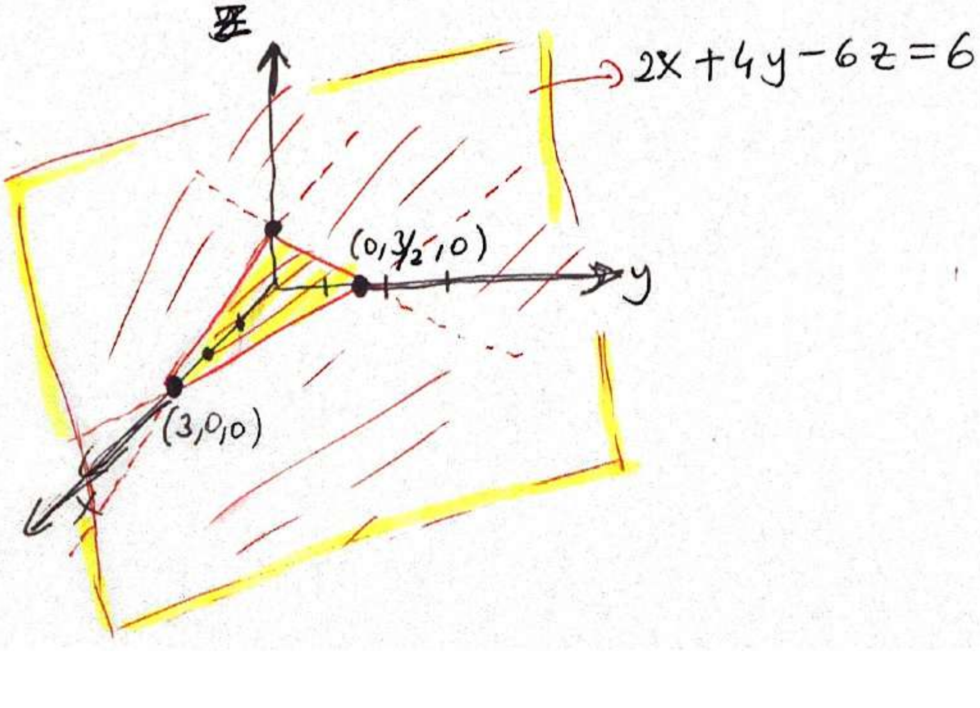
Graphing Planes:

Ex. Graph the planes given by

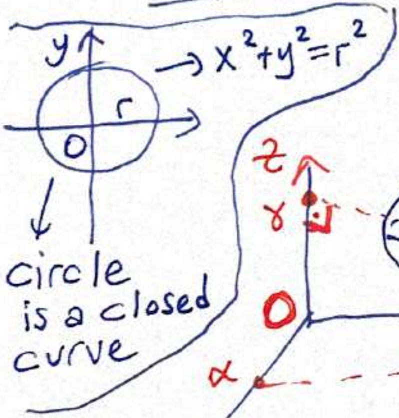
$x=3$, $x+y=2$, $2x+4y-6z=6$



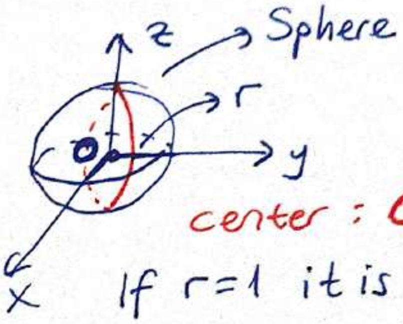




Spheres:

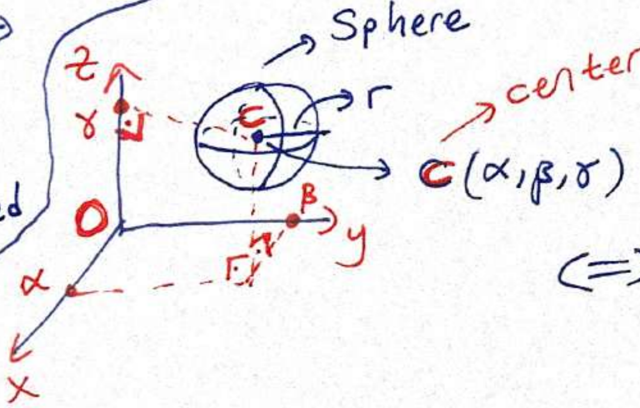


$$x^2+y^2+z^2=r^2 \iff$$



center : $O(0,0,0)$

If $r=1$ it is unit sphere.



$$\iff (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$$

Standart form of the eq. of a sphere.

Ex. Show that the graph of the eq.

$$x^2+y^2+z^2-2x+6z+1=0$$

is a sphere and find its radius and center.

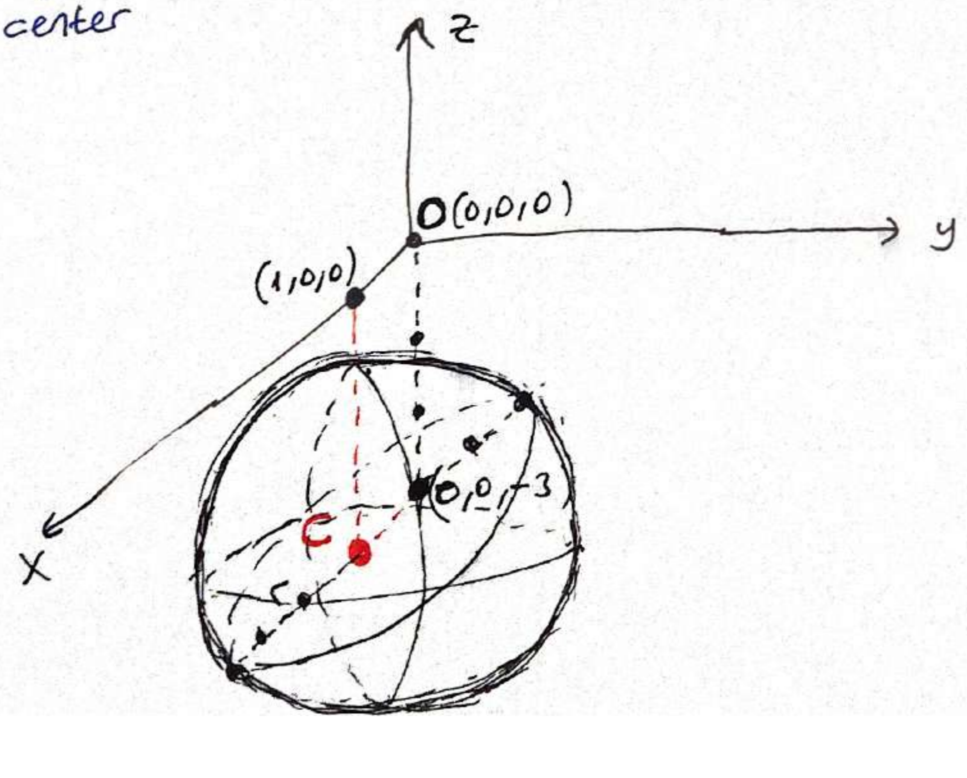
$$(x^2-2x)+y^2+(z^2+6z)+1=0$$

$$(x^2-2x+1^2)+y^2+(z^2+6z+3^2)=-1+1^2+3^2$$

$$(x-1)^2+y^2+(z+3)^2=9$$

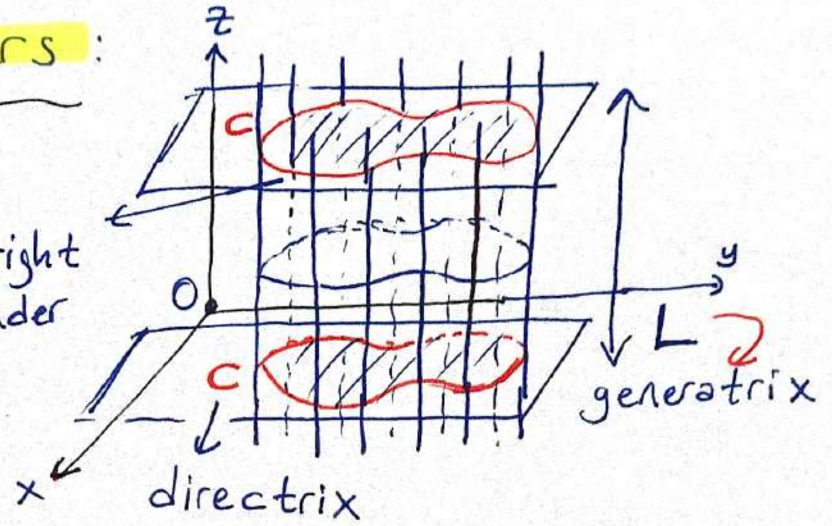
$C(1,0,-3)$, $r=3$.

center $\qquad$ radius

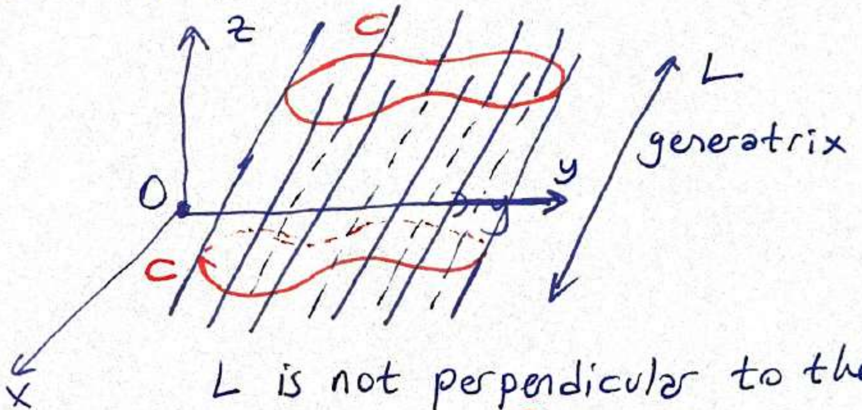


Cylinders:

The right cylinder



The L is perpendicular to the plane containing C.



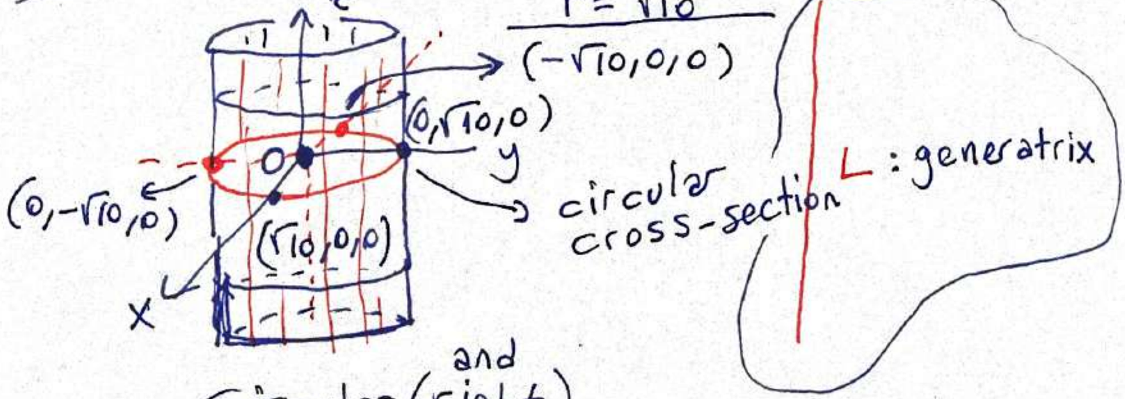
L is not perpendicular to the plane containing C.

In 3D draw (sketch) the following:

Ex: $x^2+y^2=10$

$r^2=10$

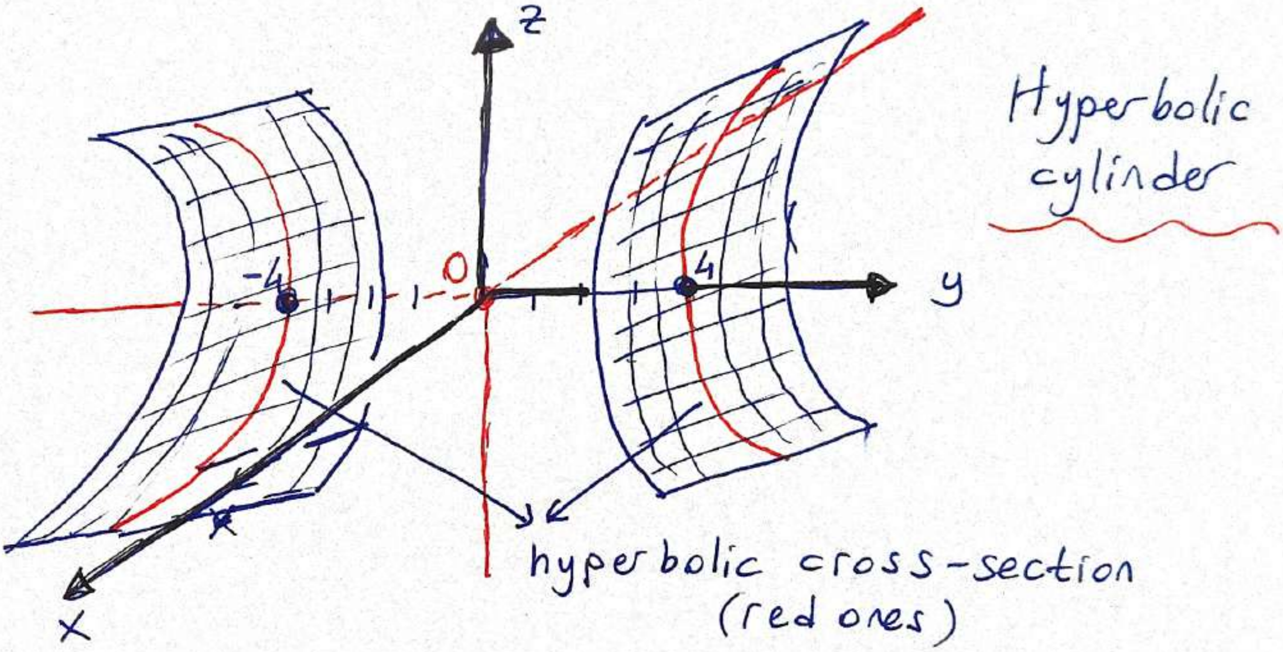
$r=\sqrt{10}$



Circular (right) and cylinder

Ex: $y^2 - z^2 = 16$

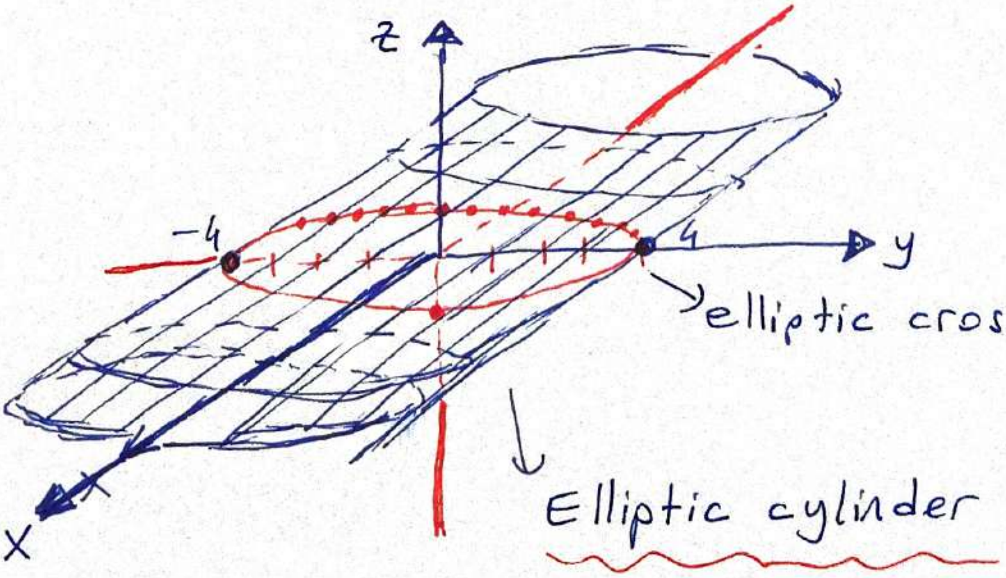
- $y=0 \Rightarrow z^2 = -16 \Rightarrow$ surface does not intersect with axis z.
- $z=0 \Rightarrow y^2 = 16 \Rightarrow y = \mp 4$



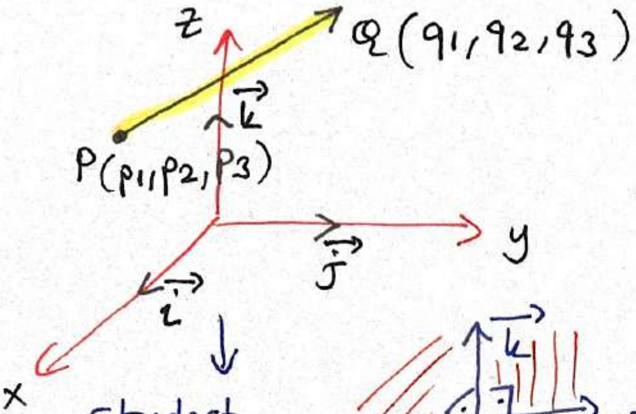
Ex: $y^2 + 9z^2 = 16$

$y=0 \Rightarrow z^2 = \frac{16}{9} \Rightarrow z = \mp \frac{4}{3}$

$z=0 \Rightarrow y^2 = 16 \Rightarrow y = \mp 4$

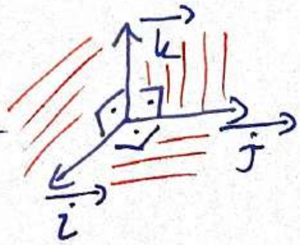


Vectors in $\mathbb{R}^3$:



$$\overrightarrow{PQ} = Q - P$$
$$= \langle q_1 - p_1,\ q_2 - p_2,\ q_3 - p_3 \rangle$$

standart vector basis of $\mathbb{R}^3$.



$\vec{i} = \langle 1, 0, 0 \rangle$

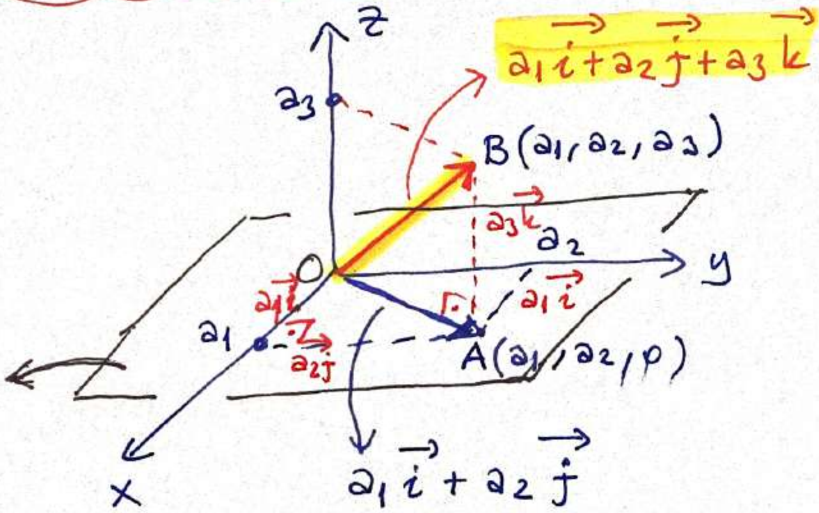
$j = \langle 0, 1, 0 \rangle$

$k = \langle 0, 0, 1 \rangle$

$$\|\vec{i}\| = \sqrt{1^2 + 0^2 + 0^2} = 1, \quad \|\vec{j}\| = 1, \quad \|\vec{k}\| = 1.$$

$$\overrightarrow{PQ} = \langle q_1 - p_1,\ q_2 - p_2,\ q_3 - p_3 \rangle$$
$$\overrightarrow{PQ} = (q_1 - p_1)\vec{i} + (q_2 - p_2)\vec{j} + (q_3 - p_3)\vec{k} \quad \text{---}(*)$$

standart representation



plane xy or xOy

In $(*)$, $\|\overrightarrow{PQ}\| = ?$

$$\|\overrightarrow{PQ}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

Ex: Find the standart representation of $\overrightarrow{PQ}$ with starting point $P(2,3,-1)$ and finishing point $Q(1,-4,2)$, and also $\|\overrightarrow{PQ}\| = ?$

$$\overrightarrow{PQ} = (1-2)\vec{i} + (-4-3)\vec{j} + (2-(-1))\vec{k}$$

$$\overrightarrow{PQ} = -\vec{i} - 7\vec{j} + 3\vec{k}.$$

$$\|\overrightarrow{PQ}\| = \sqrt{(-1)^2+(-7)^2+3^2} = \sqrt{1+49+9} = \sqrt{59}.$$

$\downarrow$

{ distance between points P and Q. }

If $\vec{v}$ is a defined non-zero vector in $\mathbb{R}^3$, then the unit vector $\vec{u}$ that points in the same direction as $\vec{v}$ is obtained by $\boxed{\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}}$ (as in $\mathbb{R}^2$).

$$\vec{u} = \frac{1}{\|\vec{v}\|} \cdot \vec{v}$$

$\rightarrow$ positive scalar

$\boxed{\|\vec{v}\| \neq 0}$

$$u = m \cdot v$$
$$\downarrow$$
$$m > 0$$

Ex: Find the unit vector that points in the direction of the vector $\overrightarrow{AB}$ from $A(1,2,3)$ to $B(-1,0,4)$.

$$\overrightarrow{AB} = (-1-1)\vec{i} + (0-2)\vec{j} + (4-3)\vec{k}$$

$$\overrightarrow{AB} = -2\vec{i} - 2\vec{j} + \vec{k}$$

$$\vec{u} = \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{-2\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{(-2)^2+(-2)^2+1^2}} = \frac{-2\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{9}}$$

$$= -\frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}.$$

Ch. 9.3. The Dot Product

A product of two vectors that yields a scalar is called **dot** (= inner = scalar) **product**.

The dot product of $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is determined by

$$\underbrace{\vec{u} \bullet \vec{v}}_{\text{dot product notation}} = u_1v_1 + u_2v_2 + u_3v_3 \in \mathbb{R}.$$

Ex: Find the product of $\vec{u} = -\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = 2\vec{i} - 3\vec{j} + 4\vec{k}$.

$\vec{u} \bullet \vec{v} = (-1)2 + 2(-3) + 3 \cdot 4 = -2 - 6 + 12 = 4 \in \mathbb{R}$.

Properties of dot product:

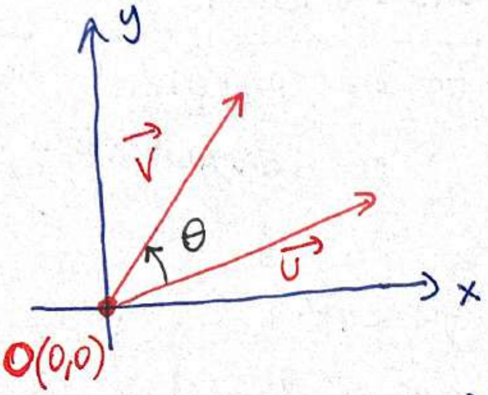
$\forall\ \vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $\forall c \in \mathbb{R}$,

1) $\vec{v} \bullet \vec{v} = \|\vec{v}\|^2$, (Magnitude of a vector)
2) $\vec{0} \bullet \vec{v} = 0$, (zero product)
3) $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$ (commutativity)
4) $c(\vec{v} \bullet \vec{w}) = (c\vec{v}) \bullet \vec{w} = \vec{v} \bullet (c\vec{w})$, (scalar multiple)
5) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$ (distributivity)

Proof. 1) $\|\vec{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = v_1^2 + v_2^2 + v_3^2$

$= \vec{v} \bullet \vec{v}$.

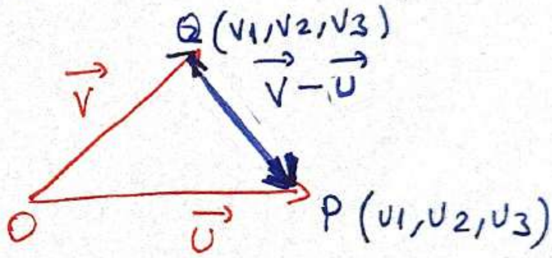
Angle Between Two Vectors:



If θ is the angle between the non-zero $\vec{u}$ and $\vec{v}$, then

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}}$$

Proof: Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$,



Consider ΔAOB:

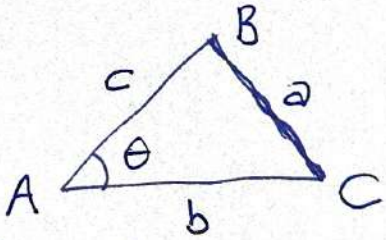
Then, $\|\overrightarrow{OP}\| = \|\vec{u}\| = \sqrt{u_1^2+u_2^2+u_3^2}$,

$\|\overrightarrow{OQ}\| = \|\vec{v}\| = \sqrt{v_1^2+v_2^2+v_3^2}$,

$\|\overrightarrow{QP}\| = \|\vec{v}-\vec{u}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2}$.

Using the law of cosines:

$a^2 = b^2+c^2-2bc\cos\theta$



$$\|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$$

$$\cos\theta = \frac{\|\vec{u}\|^2+\|\vec{v}\|^2-\|\vec{v}-\vec{u}\|^2}{2\|\vec{u}\|\cdot\|\vec{v}\|}$$

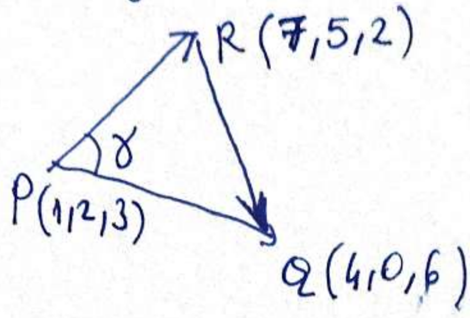
$$= \frac{u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2-\left[(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2\right]}{2\|\vec{u}\|\cdot\|\vec{v}\|}$$

$$= \big(u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2-u_1^2+2u_1v_1-u_2^2$$
$$-v_2^2+2u_2v_2-v_2^2-u_3^2+2u_3v_3-v_3^2\big)\big/\big(2\|\vec{u}\|\cdot\|\vec{v}$$

$$= \frac{2u_1v_1+2u_2v_2+2u_3v_3}{2\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}.\quad \text{q.e.d.}$$

→ gamma

Ex. Find the angle γ formed at P:



$$\overrightarrow{PQ} = Q - P = \langle 4-1, 0-2, 6-3 \rangle = \langle 3, -2, 3 \rangle ,$$

$$\overrightarrow{PR} = R - P = \langle 7-1, 5-2, 2-3 \rangle = \langle 6, 3, -1 \rangle ,$$

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 3.6 + (-2)3 + 3(-1) = 18 - 6 - 3 = 9$$

$$\|\overrightarrow{PQ}\| = \sqrt{3^2 + (-2)^2 + 3^2} = \sqrt{9+4+9} = \sqrt{22} ,$$

$$\|\overrightarrow{PR}\| = \sqrt{6^2 + 3^2 + (-1)^2} = \sqrt{36+9+1} = \sqrt{46}$$

$$\cos \angle QPR = \cos\gamma = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\| \cdot \|\overrightarrow{PR}\|}$$

$$\cos\gamma = \frac{9}{\sqrt{22} . \sqrt{46}} = \frac{9}{\sqrt{1042}}$$

$$\gamma = \cos^{-1}\left(\frac{9}{\sqrt{1042}}\right) = \arccos\left(\frac{9}{\sqrt{1042}}\right).$$

$$\cos\gamma \cong \frac{9}{31.8} \cong 0.283.$$

Geometrical Formula of the Dot Product:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| . \|\vec{v}\| . \cos\theta .$$

Here, θ denotes the angle $\left(\theta \in [0, \pi]\right)$ between $\vec{u}$ and $\vec{v}$.

Two vectors are called perpendicular (or orthogonal) if the angle between them is $\theta = \frac{\pi}{2}$ (or 90°)

Orthogonal vector theorem:

$\forall\, 0 \neq \vec{u}, \vec{v}$; $\vec{u}$ and $\vec{v}$ are orthogonal if and only if (iff) $\vec{u} \cdot \vec{v} = 0$.

$\downarrow$

{ For every non-zero vectors $\vec{u}$ and $\vec{v}$ }

That is, $\boxed{\vec{u} \perp \vec{v} \overset{\text{iff}}{\Longleftrightarrow} \vec{u} \cdot \vec{v} = 0}$ { zero, not zero vector }

Ex: Find $\vec{i} \cdot \vec{j}$, $\vec{i} \cdot \vec{k}$, $\vec{j} \cdot \vec{k}$ in $\mathbb{R}^3$.

$\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$;

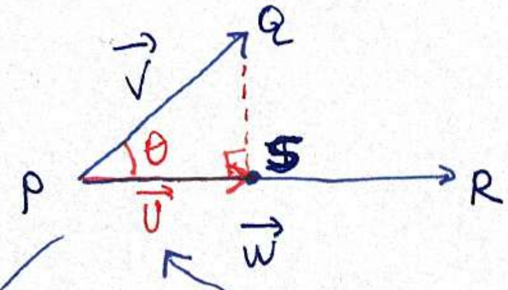
$$\vec{i} \cdot \vec{j} = \langle 1, 0, 0 \rangle \cdot \langle 0, 1, 0 \rangle$$
$$= 1.0 + 0.1 + 0.0 = 0 .$$

So $\vec{i} \perp \vec{j}$. Similarly, $\vec{i} \cdot \vec{k} = 0$, $\vec{j} \cdot \vec{k} = 0$, and then $\vec{i} \perp \vec{k}$, $\vec{j} \perp \vec{k}$.

{ $\perp$: orthogonal notation }

Projections : Let v, w be two vectors

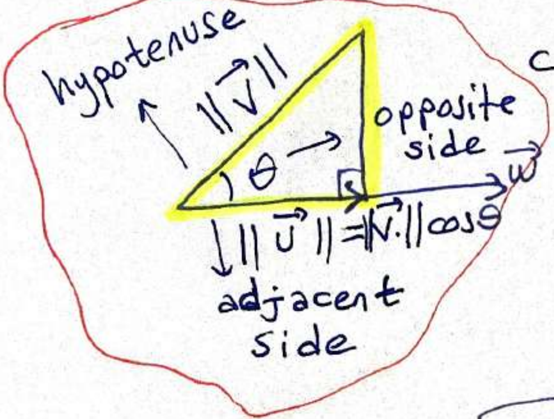
as in Figure below.
shown



When we drop a perpendicular from Q to the line described by $\vec{w}$, we obtain a vector named the vector projection of $\vec{v}$ onto $\vec{w}$. (labeled $\vec{u}$)

{ Even though that figure is drawn in $\mathbb{R}^2$ the following projection formula also applies to $\mathbb{R}^3$. }

The scalar projection of $\vec{v}$ onto $\vec{w}$ (also named component of $\vec{v}$ along $\vec{w}$) is the length of the vector projection, denoted by $\|\vec{u}\|$.



$$\cos\theta = \frac{\|\vec{u}\|}{\|\vec{v}\|} \qquad (\|\vec{v}\| \neq 0)$$

$$\|\vec{u}\| = \|\vec{v}\| \cdot \cos\theta$$

$$= \|\vec{v}\| \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|} \right)$$

$$\boxed{\|\vec{u}\| = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}.} \longrightarrow \text{scalar projection } \vec{v} \text{ onto } \vec{w}$$

$$\vec{u} = \left(\|\vec{v}\| \cos\theta \right) \frac{\vec{w}}{\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} \cdot \left(\frac{\vec{w}}{\|\vec{w}\|} \right)$$

$$\vec{u} = \frac{\vec{v} \cdot \vec{w}}{\|w\|^2} \cdot \vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \cdot \vec{w} \longrightarrow \text{vector projection } \vec{v} \text{ onto } \vec{w}$$

If $\vec{v}, \vec{w} \neq \vec{0}$, then the vector projection of $\vec{v}$ in the direction $\vec{w}$ is given by

{ projection $\vec{v}$ onto $\vec{w}$ } ← $\boxed{\mathrm{proj}_{\vec{w}} \vec{v} = \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \cdot \vec{w}}$

If $\theta \longrightarrow$ is obtuse { $90° < \theta \leq 180°$  }

its cosine is negative and then

$\|\vec{u}\| = - \|\vec{v}\| \cos\theta$

it is the scalar component of $\vec{v}$ in the direction of $\vec{w}$.

If $\theta \longrightarrow$ is acute, {  $0° \leq \theta < 90°$ } then the vector projection

has length $\|\vec{v}\| \cos\theta$ and has direction $\vec{w} / \|\vec{w}\|$ (this is the unit vector of $\vec{w}$)

If the angle is obtuse, projection has length $-\|\vec{v}\| \cos\theta$ and has direction $-\vec{w}/\|\vec{w}\|$.

Scalar projection of $\vec{v}$ onto $\vec{w}$ is obtained by

(a number) $\boxed{\mathrm{comp}_{\vec{w}} \vec{v} = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}}$

$$\mathrm{comp}_{\vec{w}} \vec{v} \begin{cases} > 0, & \text{if } \theta \in [0, \frac{\pi}{2}) \quad \text{(acute)} \\ < 0, & \text{if } \theta \in (\frac{\pi}{2}, \pi] \quad \text{(obtuse)} \end{cases}$$

semi open interval

Ex: Find the projections (vector and scalar) of $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$ onto $\vec{w} = -2\vec{i} + \vec{j} + \vec{k}$.

Vector projection:

$$\text{proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$$

$$= \left(\frac{1.(-2) + 2.1 + 3.1}{(-2)^2 + 1^2 + 1^2} \right) (\vec{i} + 2\vec{j} + 3\vec{k})$$

$$= \frac{-2+2+3}{6} (\vec{i} + 2\vec{j} + 3\vec{k})$$

$$= \frac{3}{6} (\vec{i} + 2\vec{j} + 3\vec{k})$$

$$= \frac{1}{2} (\vec{i} + 2\vec{j} + 3\vec{k})$$

Scalar projection:

$$\text{comp}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}$$

$$= \frac{1.(-2) + 2.1 + 3.1}{\sqrt{1^2 + 2^2 + 3^2}}$$

$$= \frac{3}{\sqrt{14}}$$

9.4 : The Cross Product :

If $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$, $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$,
the cross product (yields a vector) is determined by
the following determinant :

$$\boxed{\underbrace{\vec{u} \times \vec{v}}_{\text{cross (or vector) product notation}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}} .$$

Ex. For $\vec{u} = 5\vec{i} - \vec{j} + 2\vec{k}$, $\vec{v} = 3\vec{i} - \vec{k}$ find $\vec{u} \times \vec{v}$,
and $\|\vec{u} \times \vec{v}\|$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \overset{\oplus}{\vec{i}} & \overset{\ominus}{\vec{j}} & \overset{\oplus}{\vec{k}} \\ 5 & -1 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= \left[(-1)(-1) - 0.2\right]\vec{i} - \left[5.(-1) - 2.3\right]\vec{j} + \left[5.0 - 3(-1)\right]\vec{k}$$

$$\vec{u} \times \vec{v} = \vec{i} + 11\vec{j} + 3\vec{k} . \text{ (a vector)}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{1^2 + 11^2 + 3^2} = \sqrt{1 + 121 + 9} = \sqrt{131} . \text{ (a scalar)}$$

Geometric property of the cross product:

If $\vec{u}, \vec{v} \neq 0 \in \mathbb{R}^3$ that are not multiples of one another then $\vec{u}$ and $\vec{v}$ are orthogonal to $\underbrace{\vec{u} \times \vec{v}}_{= \vec{w}}$.

Proof: We want to show $\vec{u} \perp \underbrace{\vec{u} \times \vec{v}}_{= \vec{w}}$ (also $\vec{v} \perp \underbrace{\vec{u} \times \vec{v}}_{= \vec{w}}$)

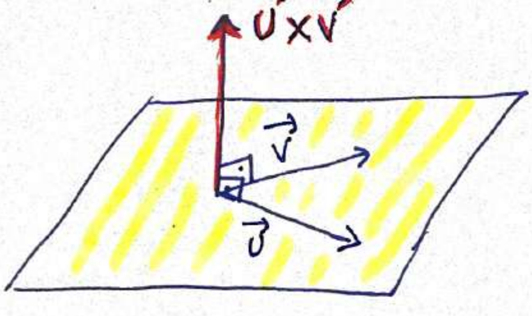
$$\vec{u} \perp \vec{w} \iff \vec{u} \cdot \vec{w} = 0$$

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \langle u_1, u_2, u_3 \rangle \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= 0. \quad \text{(Check it!)}$$

Similarly;

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$



$\vec{u} \perp \vec{u} \times \vec{v}$

and also

$\vec{v} \perp \vec{u} \times \vec{v}$

— o —

Ex. Find a nonzero vector which is perpendicular (orthogonal) to both

$\vec{u} = \vec{i} + 2\vec{j}$ and $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & -3 & 1 \end{vmatrix} = [2.1 - (-3).0]\,\vec{i} - [1.1 - 2.0]\,\vec{j} + [1.(-3) + 2.2]\,\vec{k}$$

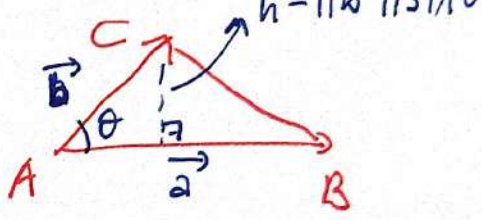
$$= 2\vec{i} - \vec{j} + \vec{k}.$$

Ex : Find the area of the triangle with vertices

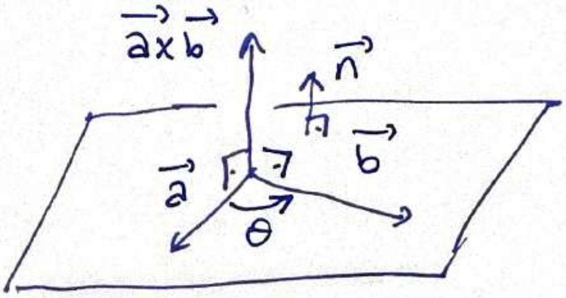
A( ), B( ), C( )

$$\vec{a} = \overrightarrow{AB} =$$
$$\vec{b} = \overrightarrow{AC} =$$
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ - & - & - \\ - & - & - \end{vmatrix}$$

$$A = \frac{1}{2} \| \vec{a} \times \vec{b} \|$$

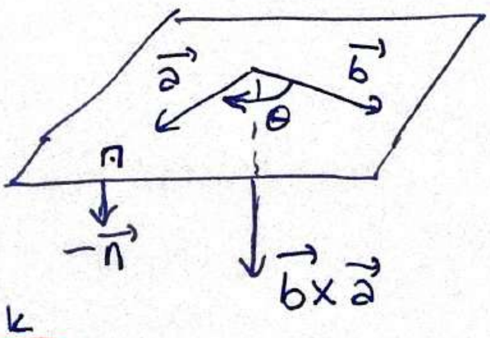


$$A = \frac{1}{2} \|\vec{a}\| . h$$
$$= \frac{1}{2} \|\vec{a}\| . \|\vec{b}\| \sin\theta$$
$$= \frac{1}{2} \|\vec{a} \times \vec{b}\|$$



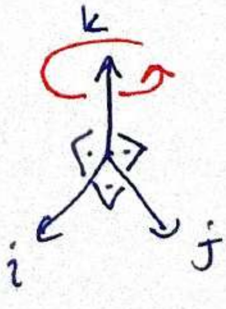
right hand rule (up)

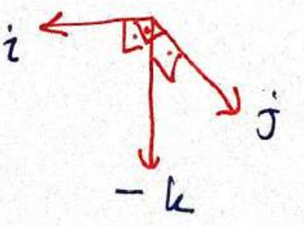
(+) positive direction



(−) negative direction

$$\boxed{\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}}$$





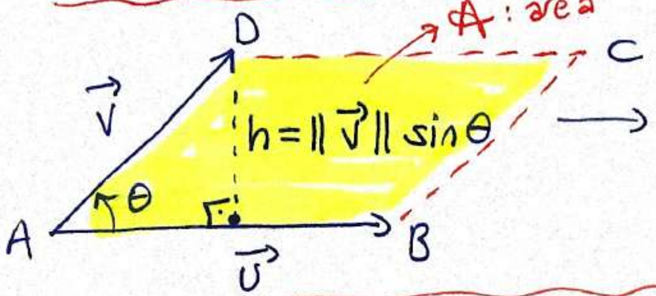
Properties of Cross Product :

Magnitude of cross product :

$\forall\ 0 \neq \vec{u}, \vec{v} \in \mathbb{R}^3$ with θ the angle between $\vec{u}$ and $\vec{v}$ $(\theta \in [0, \pi])$

{For all non-zero vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$}

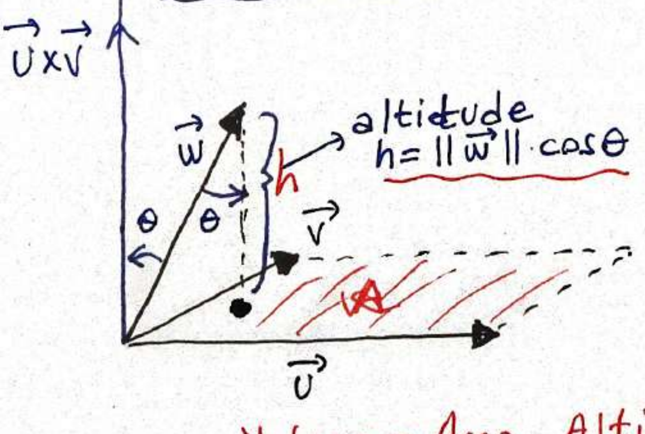
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| . \|\vec{v}\| . \sin\theta$$

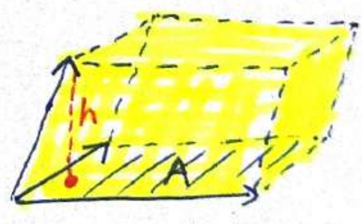


Area of parallelogram gives $\|\vec{u} \times \vec{v}\|$.

$$\boxed{\begin{aligned} \text{Area}(ABCD) &= \text{base} . \text{height} \\ &= \|\vec{u}\| . \|\vec{v}\| . \sin\theta \\ \mathcal{A} &= \|\vec{u} \times \vec{v}\| . \end{aligned}}$$

Scalar Triple Product and Volume :





$V = A . h$

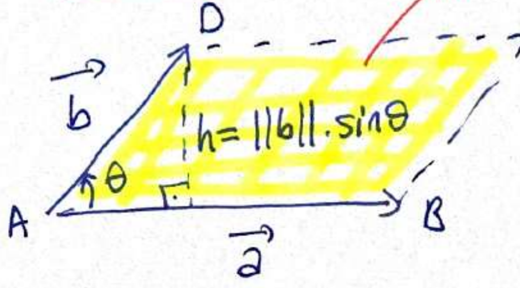
Volume = Area . Altitude

$V = \mathcal{A} . h$

{Volume of the parallelepiped}

Area :

A : area of parallelogram



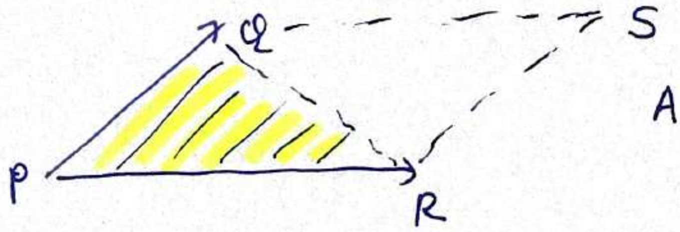
$$\text{Area}(ABCD) = \|\vec{a}\| . h$$

$$A = \|\vec{a}\| . \|b\| \sin\theta$$

Ex Find the area of $\overset{\triangle}{PQR}$ when $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$.

$$\vec{PQ} \times \vec{PR} = \langle -40, -15, 15 \rangle .$$

$$\|\vec{PQ} \times \vec{PR}\| = \sqrt{(-40)^2 + (-15)^2 + (15)^2} = 5\sqrt{82}$$

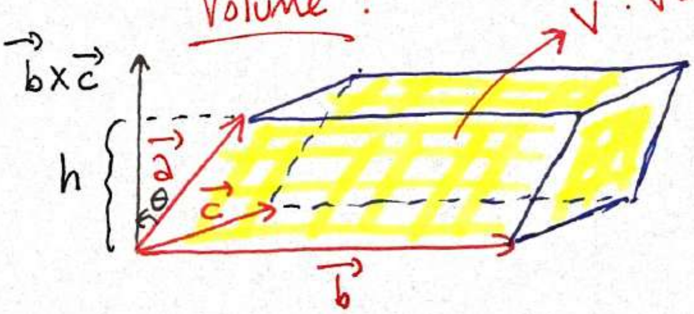


$$\text{Area}(\overset{\triangle}{PQR}) = \frac{1}{2} \text{Area}(PQSR)$$

$$= \frac{1}{2} . 5\sqrt{82} .$$

Volume :

V : volume of parallelpiped



$$\text{Volume} = \|\vec{b} \times \vec{c}\| . h$$

$$= \|\vec{b} \times \vec{c}\| . \|\vec{a}\| . \cos\theta$$

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Triple Vector Product :

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \|\vec{u}\times\vec{v}\|\cdot\|\vec{w}\|\cos\theta$$

θ: angle between $\vec{u}\times\vec{v}$ and $\vec{w}$

Altitude of parallelpiped is determined by

$$h = \left|\,\|\vec{w}\|\cos\theta\,\right| = \left|\frac{(\vec{u}\times\vec{v})\cdot\vec{w}}{\|\vec{u}\times\vec{v}\|}\right|,$$

then volume is described by

$$V = A\cdot h = \|\vec{u}\times\vec{v}\|\cdot\left|\frac{(\vec{u}\times\vec{v})\cdot\vec{w}}{\|\vec{u}\times\vec{v}\|}\right|$$

$$\boxed{V = \left|(\vec{u}\times\vec{v})\cdot\vec{w}\right|},$$

$$\boxed{V = \left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|}$$

absolute value of the determinant.

$\vec{u} = \langle u_1, u_2, u_3\rangle$,

$\vec{v} = \langle v_1, v_2, v_3\rangle$,

$\vec{w} = \langle w_1, w_2, w_3\rangle$.

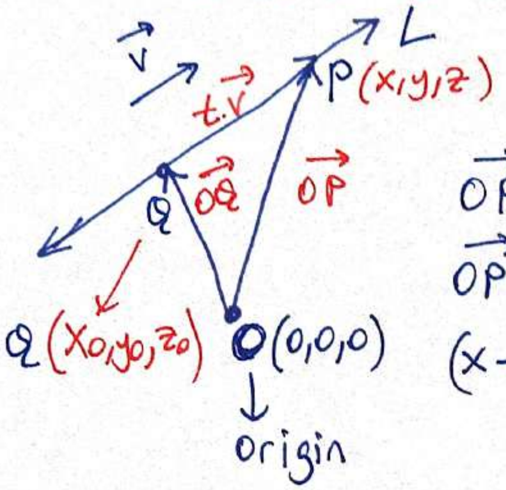
For non-zero $\vec{u},\vec{v},\vec{w}$ in $\mathbb{R}^3$,

$A = \|\vec{u}\times\vec{v}\|$ (Area of parallelogram)

$A = \frac{1}{2}\|\vec{u}\times\vec{v}\|$ (Area of triangle)

$V = \left|(\vec{u}\times\vec{v})\cdot\vec{w}\right|$ (Volume of a parallelepiped).

Ch 9.5. Lines in $\mathbb{R}^3$:



is aligned with L.

$\vec{V} \parallel L$, $\vec{v} = \langle A, B, C \rangle$

$$\vec{OP} = \vec{OQ} + \vec{QP}$$

$$\vec{OP} = \vec{OQ} + t\cdot\vec{V} \quad (t \in \mathbb{R})$$

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t\left[A\vec{i} + B\vec{j} + C\vec{k}\right]$$

Then,

$$x - x_0 = tA, \quad y - y_0 = tB, \quad z - z_0 = tC$$

$$\left.\begin{array}{l} x = x_0 + At \\ y = y_0 + Bt \\ z = z_0 + Ct \end{array}\right\}$$

Parametric form of a line in 3D.

Eliminating t in that system we get ;

$$L : \boxed{\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}},$$

symmetric eqs. for a line in $\mathbb{R}^3$

where L contains the point $Q(x_0, y_0, z_0)$ and is aligned with $\vec{v} = \langle A, B, C \rangle$ $(A, B, C \in \mathbb{R}^+)$.

Ex : Find the symmetric equations for the line L through the points $P(1, 2, 4)$ and $Q(2, -1, 3)$.

$$\vec{PQ} = (2-1)\vec{i} + (-1-2)\vec{j} + (3-4)\vec{k}$$

$$\vec{PQ} = \vec{i} - 3\vec{j} - \vec{k}.$$

The direction numbers of L are $[1, -3, -1]$. We can take either P or Q as (x_0, y_0, z_0). We take P :

$$\frac{x - 1}{1} = \frac{y - 2}{-3} = \frac{z - 4}{-1}$$

xy-plane ; $z=0 \Rightarrow \frac{x-1}{1} = \frac{-4}{1} \Rightarrow x=-3$,

$\frac{y-2}{-3} = -4 \Rightarrow y-2 = 12 \Rightarrow y = 14$,

So; $(-3, 14, 0)$ is the point of the intersection of the line with the xy-plane.

EX. Find the parametric and symmetric equations for the line through the points $P(2,4,-3)$ and $Q(3,-1,1)$. Find the intersection point of it on xy-plane.

$\overrightarrow{PQ} \uparrow\uparrow \vec{v}$, $\vec{v} = \langle 3-2, -1-4, 1-(-3) \rangle$

$= \langle 1, -5, 4 \rangle$

$= \langle A, B, C \rangle$

We chose $P_0(2,4,-3) = P$

$$\left.\begin{array}{l} A=1 \\ B=-5 \\ C=4 \end{array}\right\} \quad \left.\begin{array}{l} x = 2+t \\ y = 4-5t \\ z = -3+4t \end{array}\right\} \text{ parametric eqs. of the line.}$$

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4} \quad \text{symetric eqs.}$$

on xy-plane $z=0. \Rightarrow \frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$

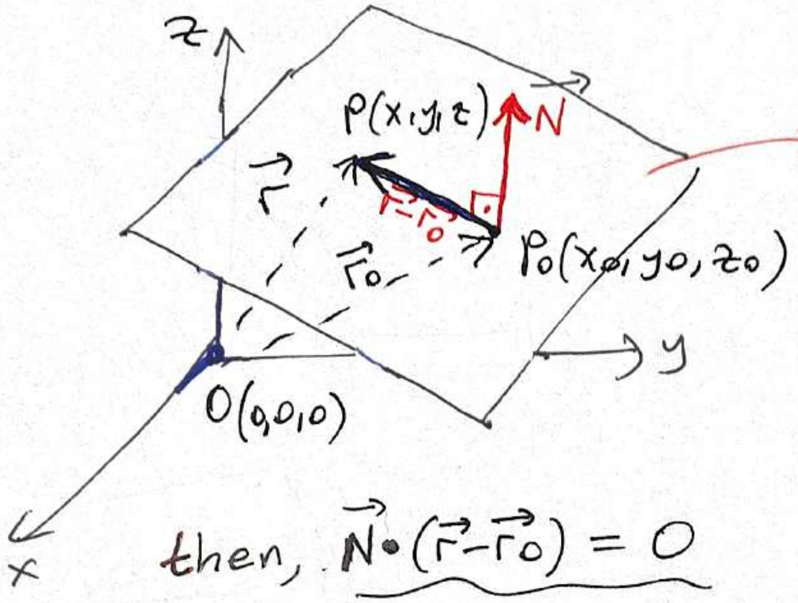
$$\left.\begin{array}{l} x-2 = \frac{3}{4} \Rightarrow x = \frac{11}{4} \\ \frac{y-4}{-5} = \frac{3}{4} \Rightarrow y = \frac{1}{4} \end{array}\right\} \Rightarrow$$

So, intersection point on xy-plane is

$\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Ch. 9.6 Planes in $\mathbb{R}^3$

$N = \langle A, B, C \rangle$



→ Plane

$\vec{N} \perp (\vec{r}-\vec{r}_0)$

then, $\vec{N} \cdot (\vec{r}-\vec{r}_0) = 0$

$$\langle A, B, C \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

Scalar eq. of the plane.

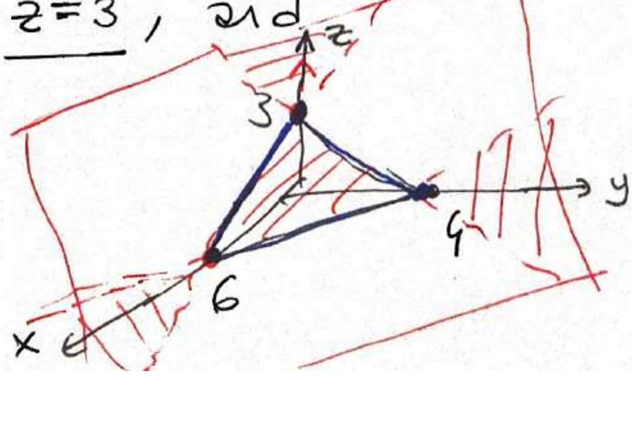
Ex Find the scalar eq of the plane passing through $P_0(2,4,-1)$ and has normal vector $\vec{N} = \langle 2,3,4 \rangle$.

$$\left.\begin{aligned} A &= 2 \\ B &= 3 \\ C &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} &2(x-2) + 3(x-4) + 4(z+1) = 0 \\ &2x + 3y + 4z = 12 \end{aligned}$$

It intersects the axis x : taking $y = z = 0$ then $\Rightarrow$ $x = 6$.

Similarly ; $x = y = 0 \Rightarrow z = 3$, and

$x = z = 0$ then $y = 4$.



$U \cdot V = \|U\| \cdot \|\vec{V}\| \cdot \cos\theta$

$\theta = 0°$: $\uparrow\uparrow$	$\theta = 90°$	$\theta = 180°$ $\downarrow\uparrow$
$\vec{U} \cdot \vec{V} = \|\vec{U}\| \cdot \|\vec{V}\|$	$\vec{U} \cdot \vec{V} = 0$	$\vec{U} \cdot \vec{V} = -\|\vec{U}\| \cdot \|\vec{V}\|$

$\vec{U} \cdot \vec{V} = V \cdot \vec{U}$ commutative

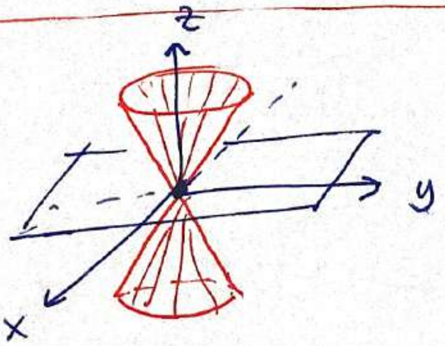
$\vec{U} \times \vec{V} = -\vec{V} \times \vec{U}$ anti-commutative

— o —

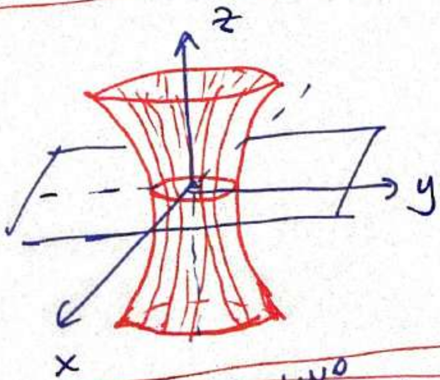
Ch. 9.7 Quadric Surfaces :

Elliptic Cone :

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



($a = b = r$
Circular cone)

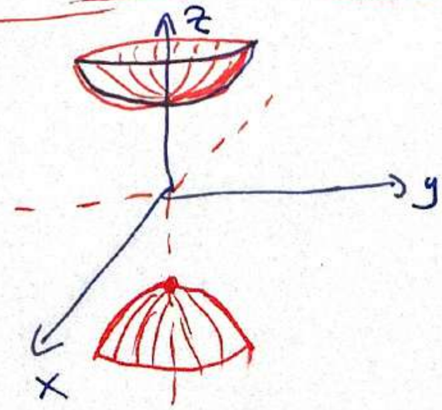


Hyperboloid of one sheet

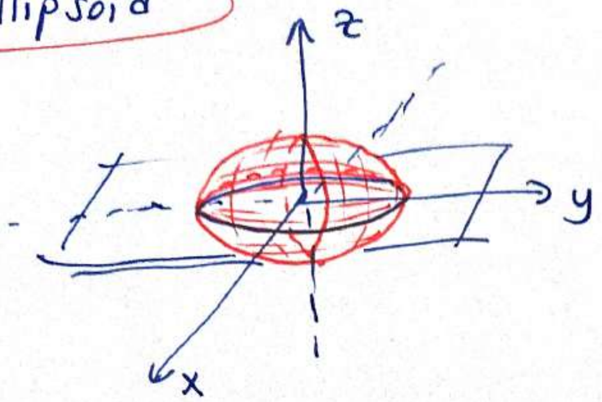
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



Ellipsoid

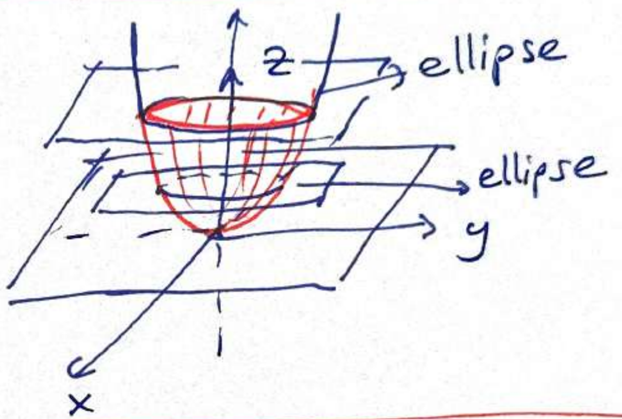


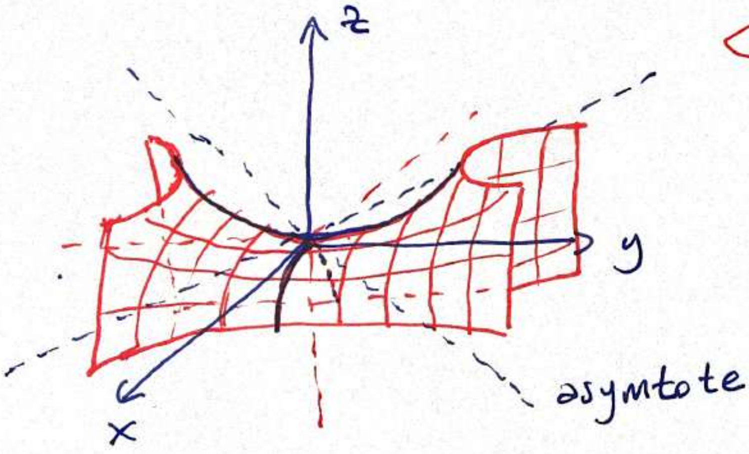
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$\left(\begin{array}{c} a=b=c=r \\ \text{sphere} \end{array} \right)$

Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$





Hyperbolic paraboloid

$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

Ex. Show that lines L_1 and L_2 are skew lines.

L_1: $x = 1 + t$, $y = -2 + 3t$, $z = 4 - t$,

L_2: $x = 2k$, $y = 3 + k$, $z = -3 + 4k$.

$$\left.\begin{array}{l}\vec{v}_1 = \langle 1, 3, -1 \rangle \\ \vec{v}_2 = \langle 2, 1, 4 \rangle\end{array}\right\} \quad \vec{v}_1 \neq m . \vec{v}_2 .$$

Then, L_1 and L_2 are not parallel lines.

If they intersect in a point, the lines must have common solutions t and k. Hence,

$$\left.\begin{array}{ll}(1) & 1 + t = 2k \\ (2) & -2 + 3t = 3 + k \\ (3) & 4 - t = -3 + 4k\end{array}\right\} \text{from first two eqs. we get :}$$

$$\underset{(1)}{t = 2k - 1} \Rightarrow \underset{(2)}{-2 + 3(2k-1) = 3 + k}$$

$$\left.\begin{array}{l} \Rightarrow -2 + 6k - 3 = 3 + k \\ \Rightarrow 6k - k = 3 + 3 + 2 \\ \Rightarrow 5k = 8 \\ \Rightarrow \underline{\underline{k = \frac{8}{5}}} . \end{array}\right\}$$

Substituting $k = \frac{8}{5}$ into (1) or (2), for example into (1):

$$1 + t = 2 . \left(\frac{8}{5}\right) = \frac{16}{5}$$

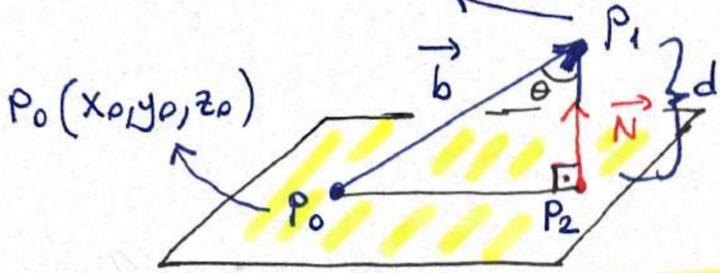
$$\underline{\underline{t = 11/5}}$$

Eq. (3) does not supply for k and t :

$$4 - \left(\frac{11}{5}\right) \stackrel{?}{=} -3 + 4 . \left(\frac{8}{5}\right) \Rightarrow \frac{9}{5} \neq \frac{17}{5} .$$

Hence, L_1 and L_2 are skew lines.

Ex. Find the distance between point $P_1(x_1, y_1, y_1)$ and plane $Ax+By+Cz+D=0$.



$|P_1P_2| = d = ?$ distance

$$\boxed{d = \left|\operatorname{comp}_{\vec{N}} \vec{b}\right| = \frac{|\vec{N} \cdot \vec{b}|}{\|\vec{N}\|}}$$

absolute value

magnitude

{ scalar projection of $\vec{b}$ onto $\vec{N}$ }

$$\Rightarrow d = \frac{|A(x_1-x_0) + B(y_1-y_0) + C(z_1-z_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$\Rightarrow d = \frac{|(Ax_1 + By_1 + Cz_1) - (Ax_0 + By_0 + Cz_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$\Rightarrow \boxed{d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}}.$$

Here, $D = -(Ax_0 + By_0 + Cz_0)$ since P_0 on the plane.

Ex: Find the distance between parallel lines
$10x+2y-2z=5$ and $5x+y-z=1$.

$$\left.\begin{array}{l}\vec{N_1}=\langle 10,2,-2\rangle \\ \vec{N_2}=\langle 5,1,-1\rangle\end{array}\right\} \Rightarrow \vec{N_1}=2\vec{N_2}$$

$\Rightarrow \vec{N_1}$ and $\vec{N_2}$ are parallel.

Hence the planes are also parallel.
on plane $10x+2y-2x=5$ taking $y=z=0$,
then $x=\frac{1}{2}$. So, $P(\frac{1}{2},0,0)$ on that plane.

Hence, distance between point P and plane
$5x+y-z-1=0$ is determined by

$$d=\frac{\left|5.(\frac{1}{2})+1.(0)-1.(0)-1\right|}{\sqrt{5^2+1^2+(-1)^2}}$$

$$\Rightarrow d=\frac{\frac{5}{2}-1}{\sqrt{25+1+1}}=\frac{\frac{3}{2}}{3\sqrt{3}}=\frac{\sqrt{3}}{6}.$$